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Nonlinear wave propagation through a stratified atmosphere

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Abstract

We examine the propagation of sound waves through a stratified atmosphere. The method of multiple scales is employed to obtain an asymptotic equation which describes the evolution of sound waves in an atmosphere with spatially dependant density and entropy fields. The evolution equation is an inviscid Burger-like equation which contains quadratic and cubic nonlinearities, and a curvature term all of which are functions of the space variables. A model equation is derived when the modulations of the signal in a direction transverse to the direction of propagation become significant.

Keywords: Hyperbolic system; Mixed nonlinearity; Singular rays

1. Introduction

The effectiveness of the geometrical optics method prompted many workers to extend the underlying ideas to wave motion (see, for example, Choquet-Bruhat [1], Hunter and Keller [2], Majda and Rosales [3], Cramer and Sen [4], Kluwick and Cox [5], and Srinivasan and Sharma [6]); the technique involves the introduction of slow and fast variables and the phase functions. These multiple scale expansions have been successfully employed

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in the context of ODEs [7], where these were introduced by Krylov–Bogoliubov as a variant of the methods employed earlier by Poincaré and Lindstedt to eliminate secular terms in the perturbation expansions of Celestial Mechanics. The precise scaling of the fast variables in comparison to the slow ones may vary depending on the problem under study. Recently, Kluwick and Cox [5] have considered fast variables with a different scaling than were hitherto considered; the need for this arose from the study of systems with mixed nonlinearities where the nonlinear distortions of the disturbance make their appearance noticeable over time scales of order $O(\varepsilon^{-2})$. Practically, any problem of acoustics takes place in the presence of a gravitational field, and as a consequence, the unperturbed state is not uniform. In problems of propagation over large distances in atmosphere or the ocean, these effects may be crucially important and produce amplification and refraction of the sound waves. Here, using the analytical apparatus developed by Kluwick and Cox [5], we study a quasilinear system with a source term that describes the propagation of sound waves in an atmosphere with spatially dependent density and entropy fields, and derive the transport equation for the high frequency wave amplitude in the leading order terms in the expansion. The quadratic and cubic nonlinear terms in the evolution equation strongly predict that smooth solutions develop shocks and result in the breakdown of the solution. Finally, a model equation is derived when the modulations of the signal in directions transverse to the direction of propagation become significant.

2. Basic equations

Equations describing the propagation of sound waves through a stratified fluid may be expressed in the form:

$$\begin{aligned}
&\mathbf{v}_{,t} + (\mathbf{v}.\boldsymbol{\nabla})\mathbf{v} + \rho^{-1}\nabla p = -\mathbf{g},\\
&\rho_{,t} + (\mathbf{v}.\boldsymbol{\nabla})\rho + \rho(\nabla.\mathbf{v}) = 0,\\
&s_{,t} + (\mathbf{v}.\boldsymbol{\nabla})s = 0,
\end{aligned} \tag{1}$$

where ρ is the density of the fluid, $p = p(\rho, s)$ the pressure, s the entropy, t the time, ∇ the gradient operator with respect to the space coordinates (x_1, x_2, x_3), and $\mathbf{g}$ the forcing function, which balances the initial conditions; a comma followed by the letter t denotes partial differentiation with respect to time, t. The reference state is characterized by a flow field at rest, $\mathbf{v} = \mathbf{0}$, with spatially varying density and entropy fields, namely $\rho_0 = \rho_0(\mathbf{x})$ and $s_0 = s_0(\mathbf{x})$ with $\nabla p_0 + \rho_0\mathbf{g} = 0$, where the subscript '0' is used to characterize the unperturbed fluid in equilibrium. The governing system (1) can be cast into the form

$$\mathbf{U}_{,t} + A^k(\mathbf{U})\mathbf{U}_{,k} + \mathbf{F} = 0, \quad k = 1, 2, 3, \tag{2}$$

representing a hyperbolic quasilinear system of equations with source terms, which may be attributed to the influences of gravity. Here $\mathbf{U}$ and $\mathbf{F}$ are column vectors defined as $\mathbf{U} = (v_1, v_2, v_3, \rho, s)'$ and $\mathbf{F} = (g_1, g_2, g_3, 0, 0)'$, respectively, with a prime denoting transposition; a comma followed by an index k denotes partial derivative with respect to x_k, and A^k are 5×5 matrices defined as

$$A^k(\mathbf{U}) = \begin{bmatrix} v_k & 0 & 0 & a^2\delta_{1k}/\rho & p_{,s}\delta_{1k}/\rho \\ 0 & v_k & 0 & a^2\delta_{2k}/\rho & p_{,s}\delta_{2k}/\rho \\ 0 & 0 & v_k & a^2\delta_{3k}/\rho & p_{,s}\delta_{3k}/\rho \\ \rho\delta_{1k} & \rho\delta_{2k} & \rho\delta_{3k} & v_k & 0 \\ 0 & 0 & 0 & 0 & v_k \end{bmatrix},$$

where a is the sound speed given by $a^2 = p_{,\rho}|_s$ and δ is the Kronecker symbol.

3. Small amplitude high frequency waves

We look for a small amplitude high frequency asymptotic solution of (2) when the length L of the disturbed region is small in comparison with the scale height H of stratification, which is described as a typical value of $\rho_0|\nabla\rho_0|^{-1}$, so that $\varepsilon = L/H \ll 1$. In this limit, it is assumed that the perturbations caused by the wave are of size $O(\varepsilon)$, and they depend significantly on the fast characteristic variable $\xi = \phi(\mathbf{x}, t)/\varepsilon^2$, where ϕ is the phase function to be determined; with this change in variables, the system (2) becomes

$$\varepsilon^2\big(\mathbf{U}_{,t} + A^k\mathbf{U}_{,k} + \mathbf{F}\big) + \big(\phi_{,t}I + \phi_{,k}A^k\big)\mathbf{U}_{,\xi} = 0, \tag{3}$$

where I is the 5×5 unit matrix. We look for small amplitude high frequency wave solutions of (2) with an asymptotic approximation as $\varepsilon \to 0$ of the form:

$$\mathbf{U} = \mathbf{U_0}(\mathbf{x}) + \varepsilon\mathbf{U^{(1)}}(\xi, \mathbf{x}, t) + \varepsilon^2\mathbf{U^{(2)}}(\xi, \mathbf{x}, t) + \varepsilon^3\mathbf{U^{(3)}}(\xi, \mathbf{x}, t) + \cdots, \tag{4}$$

where $\mathbf{U_0} = (0, 0, 0, \rho_0(\mathbf{x}), \mathbf{s_0}(\mathbf{x}))$ is the known background state. We now expand A^k and $\mathbf{F}$ as a power series in ε about $\mathbf{U} = \mathbf{U_0}$, and use them in (3) together with (4) so that the resulting asymptotic expansion becomes

$$\varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + \cdots = 0; \tag{5}$$

we recall here that $A_0^k\mathbf{U}_{0,k} + \mathbf{F_0} = 0$; A_0^k and $\mathbf{F_0}$ being the reference values of A^k and $\mathbf{F}$, respectively. It follows immediately that to the first order, $\mathbf{U^{(1)}} = \sigma(\xi, \mathbf{x}, t)\mathbf{r}$, where σ is the scalar amplitude which will be determined to the next order and $\mathbf{r}$ is the right eigenvector of $\phi_{,k}A_0^k$ corresponding to an eigenvalue, λ (we denote the corresponding left eigenvector by $\mathbf{l}$). Furthermore, the linearized system associated with (3) admits five families of characteristic surfaces, two of which represent waves propagating with speeds $\pm a_0|\nabla\phi|$, through the background state $\mathbf{U} = \mathbf{U_0}$, and the remaining three form a set of coincident characteristics representing entropy waves or particle paths. Here we shall be concerned with the propagation of a right running acoustic wave, $\phi(\mathbf{x}, t) = \text{constant}$, propagating with speed $a_0|\nabla\phi|$. The left and right eigenvectors $\mathbf{l}$ and $\mathbf{r}$ associated with the eigenvalue $a_0|\nabla\phi|$, are given by

$$\mathbf{l} = \left(n_1, n_2, n_3, \frac{a_0}{\rho_0}, \frac{p_{,s0}}{\rho_0 a_0}\right), \qquad \mathbf{r} = \left(n_1, n_2, n_3, \frac{\rho_0}{a_0}, 0\right), \tag{6}$$

where n_1, n_2 and n_3 are the components of the unit normal $\mathbf{n}$ to the wavefront $\phi(\mathbf{x}, t) =$ constant. In this context, we refer to the work of Cramer and Sen [4] and Kluwick and

Cox [5], who examine the propagation of high frequency pulses when the nonlinear effects are noticeable over times of order $O(\varepsilon^{-2})$ rather than those of $O(\varepsilon^{-1})$; their main results focus on the case when the quadratic nonlinearity parameter Γ defined as

$$\Gamma = \frac{\mathbf{l}[\mathbf{r}.(\nabla_{\mathbf{U}} A^k)_0]\mathbf{r}\phi_{,k}}{(\mathbf{l}.\mathbf{r})} \tag{7}$$

is of order $O(\varepsilon)$; here $\nabla_{\mathbf{U}}$ is the gradient operator with respect to the vector $\mathbf{U}$, and

$$\mathbf{r}.\big(\nabla_{\mathbf{U}} A^k\big)_0 = \sum_{j=1}^{5} r_j \frac{\partial A^k}{\partial U_j}\bigg|_{\mathbf{U}=\mathbf{U_0}}.$$

Computation of Γ using (6) and the definition of A_0^k yields $\Gamma = a_0^{-1}\partial(a\rho)/\partial\rho|_0$. In order to account for a small but nonzero Γ, it would be convenient to write (5) as

$$\varepsilon z_1 + \varepsilon^2(z_2 - \mu) + \varepsilon^3(z_3 + \tilde{\mu}) + \cdots = 0, \tag{8}$$

where $\mu = \Gamma \mathbf{r}\sigma\sigma_{,\xi}$ and $\tilde{\mu} = \hat{\Gamma}\mathbf{r}\sigma\sigma_{,\xi}$ with $\hat{\Gamma} = \Gamma/\varepsilon = O(1)$. Thus, to the second and third order, we obtain respectively the following equations for $\mathbf{U}^{(2)}$ and $\mathbf{U}^{(3)}$:

$$\big(\phi_{,t} I + A_0^k \phi_{,k}\big)\mathbf{U}_{,\xi}^{(2)} = -\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}} A^k\big)_0\big]\mathbf{r}\sigma\sigma_{,\xi} + \Gamma\mathbf{r}\sigma\sigma_{,\xi}, \tag{9}$$

$$\begin{aligned}\big(\phi_{,t} I + A_0^k \phi_{,k}\big)\mathbf{U}_{,\xi}^{(3)} = {} & -(\sigma\mathbf{r})_{,t} - A_0^k(\sigma\mathbf{r})_{,k} - \big[\mathbf{r}.\big(\nabla_{\mathbf{U}} A^k\big)_0\big]\mathbf{U}_{,k}^{(0)}\sigma - \hat{\Gamma}\mathbf{r}\sigma\sigma_{,\xi} \\ & - \phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}} A^k\big)_0\big]\mathbf{U}_{,\xi}^{(2)}\sigma - \phi_{,k}\big[\mathbf{U}^{(2)}.\big(\nabla_{\mathbf{U}} A^k\big)_0\big]\mathbf{r}\sigma_{,\xi} \\ & - \frac{1}{2}\phi_{,k}\big[\mathbf{rr} : \big(\nabla_{\mathbf{U}}\nabla_{\mathbf{U}} A^k\big)_0\big]\mathbf{r}\sigma^2\sigma_{,\xi} - \big[\mathbf{r}.(\nabla_{\mathbf{U}}\mathbf{F})_0\big]\sigma,\end{aligned} \tag{10}$$

where $\mathbf{rr} : (\nabla_{\mathbf{U}}\nabla_{\mathbf{U}} A^k)_0 = \sum_{m,n=1}^{5} r_m r_n \frac{\partial^2 A^k}{\partial U_m \partial U_n}|_{\mathbf{U}=\mathbf{U_0}}$.

Thus, the solvability condition for $\mathbf{U}^{(2)}$, i.e., the right-hand side of (9) be orthogonal to $\mathbf{l}$, is satisfied automatically in view of the definition of Γ as in (7), whilst the solvability condition for $\mathbf{U}^{(3)}$ is the desired evolution equation for σ, viz.,

$$\begin{aligned}&\frac{d\sigma}{dt} + \left(\hat{\Gamma} + \frac{M}{2}\sigma\right)\sigma\frac{\partial\sigma}{\partial\xi} + \big(\mathbf{b}.\mathbf{U}^{(2)}\big)\frac{\partial\sigma}{\partial\xi} + \left(\mathbf{c}.\frac{\partial\mathbf{U}^{(2)}}{\partial\xi}\right)\sigma + \chi\sigma \\ &\quad + \frac{\Gamma(\mathbf{l}.\mathbf{U}^{(2)}\sigma)_{,\xi}}{2} = 0,\end{aligned} \tag{11}$$

where $d/dt = \partial/\partial t + a_0\phi_{,k}\,\partial/\partial x_k$ denotes the derivative along the ray, i.e., the trajectory of an element of the surface $\phi(\mathbf{x}, t) =$ constant, and M, $\mathbf{b}$, $\mathbf{c}$ and χ are given by

$$\begin{aligned}M &= \frac{\{\phi_{,k}\mathbf{l}.[\mathbf{rr} : (\nabla_{\mathbf{U}}\nabla_{\mathbf{U}} A^k)_0]\mathbf{r}\}}{2}, \\ \mathbf{b} &= \frac{\{\mathbf{l}[\phi_{,k}(\nabla_{\mathbf{U}} A^k)_0]\mathbf{r} - \Gamma\mathbf{l}\}}{2}, \\ \mathbf{c} &= \frac{\{\mathbf{l}[\mathbf{r}.(\nabla_{\mathbf{U}} A^k)_0]\phi_{,k} - \Gamma\mathbf{l}\}}{2}, \\ \chi &= \frac{\{\mathbf{l}A_0^k.(\partial\mathbf{r}/\partial x_k) + \mathbf{l}[\mathbf{r}.(\nabla_{\mathbf{U}} A^k)_0]\mathbf{U}_{,k}^{(0)} + \mathbf{l}[\mathbf{r}.(\nabla_{\mathbf{U}}\mathbf{F})_0]\}}{2}.\end{aligned} \tag{12}$$

It may be noticed that in (11), a knowledge of the second order approximation, $\mathbf{U}^{(2)}$ and $\mathbf{U}^{(2)}_{,\xi}$ is essentially required; this can be accomplished by noting that (9) can be integrated to yield

$$\mathbf{G}.\mathbf{U}^{(2)} = -\frac{1}{2}\big(\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}} A^k\big)_0\big]\mathbf{r} - \Gamma\mathbf{r}\big)\sigma^2,$$

where $\mathbf{U}^{(2)} = 0$ when $\mathbf{U}^{(1)} = 0$, and $\mathbf{G} = [\phi_{,t} I + A_0^k \phi_{,k}]$. In view of Eqs. (7), $(12)_2$ and $(12)_3$, we find that $\mathbf{b}$ and $\mathbf{c}$ are orthogonal to $\mathbf{r}$ and hence they lie in the linearly independent row-space of $\mathbf{G}$. If the linearly independent rows of $\mathbf{G}$ be denoted by $\mathcal{G}_\alpha$ $(\alpha = 1, 2, 3, 4)$, then

$$\mathbf{b} = \omega_\alpha \mathcal{G}_\alpha, \qquad \mathbf{c} = \delta_\alpha \mathcal{G}_\alpha, \tag{13}$$

and hence the terms $\mathbf{b}.\mathbf{U}^{(2)}$ and $\mathbf{c}.\mathbf{U}^{(2)}_{,\xi}$ in Eq. (11) can be written as

$$\mathbf{b}.\mathbf{U}^{(2)} = -\frac{\boldsymbol{\omega}}{2}\big(\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}} A_1^k\big)_0\big] - \Gamma I_1\big)\mathbf{r}\sigma^2, \tag{14}$$

$$\mathbf{c}.\mathbf{U}^{(2)}_{,\xi} = -\boldsymbol{\delta}\big(\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}} A_1^k\big)_0\big] - \Gamma I_1\big)\mathbf{r}\sigma\sigma_{,\xi}, \tag{15}$$

where A_1^k is the 4×5 matrix obtained from A^k by deleting its last row, I_1 is the 4×5 matrix obtained from I by deleting its last row, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ and $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3, \delta_4)$. Thus the evolution equation (11) becomes

$$\frac{d\sigma}{dt} + \left(\hat{\Gamma} + \frac{\gamma}{2}\sigma\right)\sigma\frac{\partial\sigma}{\partial\xi} + \chi\sigma = 0, \tag{16}$$

where

$$\gamma = M - (\boldsymbol{\omega} + 2\boldsymbol{\delta})\big(\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}} A_1^k\big)_0\big] - \Gamma I_1\big)\mathbf{r}. \tag{17}$$

We now turn to the calculation of coefficients $\hat{\Gamma}$, γ and χ. If we define $\nabla_l A^k_{ij} = (\partial A^k_{ij}/\partial U_l)|_{\mathbf{U}=\mathbf{U_0}}$ and $\nabla_{lm} A^k_{ij} = (\partial^2 A^k_{ij}/\partial U_l \partial U_m)|_{\mathbf{U}=\mathbf{U_0}}$, we find that

$$\begin{gathered}\nabla_k A^k_{ii} = 1, \qquad \nabla_4 A^k_{k4} = -\frac{a_0^2}{\rho_0^2} + \frac{2a_0}{\rho_0} a_{\rho_0}, \qquad \nabla_5 A^k_{k4} = \frac{2a_0}{\rho_0} a_{,s0}, \\ \nabla_5 A^k_{k5} = \frac{p_{,ss0}}{\rho_0}, \qquad \nabla_4 A^k_{k5} = \frac{2a_0}{\rho_0} a_{,s0} - \frac{p_{,s0}}{\rho_0^2}, \qquad \nabla_4 A^k_{4k} = 1; \\ 1 \leqslant i \leqslant 5,\ k = 1, 2, 3,\end{gathered}$$

and thus,

$$\Gamma = \left(1 + \frac{\rho_0}{a_0} a_{,\rho 0}\right)|\nabla\phi|. \tag{18}$$

As $\Gamma = O(\varepsilon)$, it is clear that the terms involving Γ and Γ^2 can be neglected for the purpose of computing γ and χ. Further, since $\nabla_{lm} A^k_{ij}$ in M appears in the evolution equation as a combination of r_j, r_l and r_m, it is easily verified that all the terms $\nabla_{lm} A^k_{ij}$ vanish except $\nabla_{44} A^k_{k4}$, which is given by

$$\nabla_{44} A^k_{k4} = \frac{2a_0^2}{\rho_0^3}(6 + \Lambda).$$

Hence, $M = (6 + \Lambda)|\nabla\phi|/a_0$ with $\Lambda = \frac{\rho_0^2}{a_0}\frac{\partial\Sigma}{\partial\rho}(\rho_0, s_0) = O(1)$ and $\Sigma = \frac{1}{\rho}\frac{\partial(a\rho)}{\partial\rho}$. From $(12)_2$ and $(12)_3$, the vectors **b** and **c** are obtained as follows:

$$\mathbf{b} = \mathbf{c} = |\nabla\phi|\left(n_1, n_2, n_3, -\frac{a_0}{\rho_0}, a_{,s0}\right).$$

Using (13), we find that

$$\omega_\alpha = \delta_\alpha = \frac{n_\alpha}{a_0}\left(\frac{a_0\rho_0 a_{,s0}}{p_{,s0}}\right), \quad \alpha = 1, 2, 3; \qquad \omega_4 = \delta_4 = \frac{1}{\rho_0}\left(\frac{a_0\rho_0 a_{,s0}}{p_{,s0}} + 1\right),$$

and hence, $(\boldsymbol{\omega} + 2\boldsymbol{\delta})(\phi_{,k}[\mathbf{r}.(\nabla_{\mathbf{U}} A_1^k)_0] - \Gamma I_1)\mathbf{r} = 6|\nabla\phi|/a_0$; taking the vector **g** as the acceleration due to gravity and using the foregoing results, we find that

$$\gamma = \Lambda|\nabla\phi|/a_0 \quad \text{and} \quad \chi = (a_0\nabla.\mathbf{n} + a_{,s0}s_{0,x_k}n_k)/2, \tag{19}$$

where $\nabla.\mathbf{n}$ is the mean curvature of the wavefront.

It may be noted that the quadratic nonlinearity coefficient $\hat{\Gamma}$ in (16) is the genuine nonlinearity coefficient of Lax [8], whilst γ, the cubic nonlinearity coefficient, characterizes the degree of material nonlinearity; the term $\chi\sigma$ in (16) accounts for the variation in wave amplitude σ due to wave interactions with the varying medium ahead and the wavefront curvature as the wave moves along the rays. When the medium ahead of the wave is uniform and the source term is absent, Eq. (16) reduces to the form obtained by Kluwick and Cox [5]; further, the case of purely one-dimensional wave propagation is recovered on setting $\chi = 0$, and yields an equation derived by Cramer and Sen [4].

4. Singular ray expansions

The geometric ray expansion derived in Section 3 breaks down when the signal variations transverse to the direction of wave propagation become significant. This is a common feature in diffraction problems, where ray theory predicts unbounded behaviour near a singular ray. In the subsequent study, we shall derive the evolution equation for a nonlinear wave propagation associated with transverse gradients.

Extensive research has been done for the case, $\Gamma = O(1)$. Taniuti [9,10] has applied the reductive perturbation method to find a solution when diffraction effects are prevalent. A perturbative approach similar to that of Taniuti [11] has been applied by Engelbrecht [12] in two space dimensions to derive the transient wave solutions. Hunter [13] has derived an asymptotic equation to describe the diffraction of a weakly nonlinear high frequency wave in a direction transverse to its rays. Kluwick and Cox [5] have introduced the concept of hypersurfaces for describing the wavefront and surfaces containing the singular rays for a medium with mixed nonlinearity.

Following Kluwick and Cox, we intend to derive the evolution equation for a density-stratified, nonisentropic flow when diffraction effects are prevalent. In addition to the fast phase variable, $\xi = \phi(\mathbf{x}, t)/\varepsilon^2$, we introduce other fast variables such as $\eta_n = \psi_n(\mathbf{x}, t)/\varepsilon$, $n = 1, 2$, that describe the modulations of the signal in directions transverse to those of the rays. The singular rays are supposed to lie in the hypersurfaces, $\psi_n(\mathbf{x}, t)$, $n = 1, 2$. We look for a progressive wave solution of the form:

$$\mathbf{U}(\mathbf{x},t)=\mathbf{U_0}(\mathbf{x})+\varepsilon\mathbf{U^{(1)}}(\xi,\eta_1,\eta_2,\mathbf{x},t)+\varepsilon^2\mathbf{U^{(2)}}(\xi,\eta_1,\eta_2,\mathbf{x},t)\\ +\varepsilon^3\mathbf{U^{(3)}}(\xi,\eta_1,\eta_2,\mathbf{x},t)+O\big(\varepsilon^4\big),\quad \varepsilon\ll 1. \tag{20}$$

Under this transformation the system (2) changes to

$$\mathbf{U}_{,\xi}\big(\phi_{,t}I+A_0^k\phi_{,k}\big)+\varepsilon\mathbf{U}_{,\eta_n}\big(\psi_{n,t}I+A_0^k\psi_{n,k}\big)+\varepsilon^2\big(\mathbf{U}_{,t}+A_0^k\mathbf{U}_{,k}+\mathbf{F}\big)=0. \tag{21}$$

Thereafter, substituting (4) and the Taylor's series expansion for A^k in (21), we obtain an equation similar to (5). Following the same steps as before, we have at the first order,

$$\mathbf{U^{(1)}}=\mathbf{r}(\mathbf{x},t)\sigma(\xi,\eta_1,\eta_2,\mathbf{x},t), \tag{22}$$

where $\mathbf{r}$ is the right null vector of $\mathbf{G}$. Adding $\varepsilon^2\mu$ on both sides of the equation obtained from (21), we have, on equating the coefficients of ε^2 and ε^3, the following:

$$\mathbf{G}.\mathbf{U}^{(2)}_{,\xi}=-[K_n\mathbf{r}]\sigma_{,\eta_n}-\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}}A^k\big)_0\big]\mathbf{r}\sigma\sigma_{,\xi}+\Gamma\mathbf{r}\sigma\sigma_{,\xi}, \tag{23}$$

$$\mathbf{G}.\mathbf{U}^{(3)}_{,\xi}=-(\sigma\mathbf{r})_{,t}-A_0^k(\sigma\mathbf{r})_{,k}-\big[\mathbf{r}.\big(\nabla_{\mathbf{U}}A^k\big)_0\big]\mathbf{U}^{(0)}_{,k}\sigma-\hat{\Gamma}\mathbf{r}\sigma\sigma_{,\xi}-K_n\mathbf{U}^{(2)}_{,\eta_n}\\ -\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}}A^k\big)_0\big]\mathbf{U}^{(2)}_{,\xi}\sigma-\phi_{,k}\big[\mathbf{U}^{(2)}.\big(\nabla_{\mathbf{U}}A^k\big)_0\big]\mathbf{r}\sigma\sigma_{,\xi}-\big[\mathbf{r}.(\nabla_{\mathbf{U}}\mathbf{F})_0\big]\sigma\\ -\psi_{n,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}}A^k\big)_0\big]\mathbf{r}\sigma\sigma_{,\eta_n}-\frac{1}{2}\phi_{,k}\big[\mathbf{rr}:\big(\nabla_{\mathbf{U}}\nabla_{\mathbf{U}}A^k\big)_0\big]\mathbf{r}\sigma^2\sigma_{,\xi}, \tag{24}$$

where the matrix K_n is given by

$$K_n=I\psi_{n,t}+A_0^k\psi_{n,k}. \tag{25}$$

At $O(\varepsilon^2)$, the solvability condition for $\mathbf{U}^{(2)}$ and Eq. (7) require the satisfaction of the following:

$$\mathbf{l}K_n\mathbf{r}=0. \tag{26}$$

At $O(\varepsilon^3)$, the scalar product of (24) with $\mathbf{l}$ yields the solvability condition for $\mathbf{U}^{(3)}$ as follows:

$$\frac{d\sigma}{dt}+\left(\hat{\Gamma}+\frac{M}{2}\sigma\right)\sigma\frac{\partial\sigma}{\partial\xi}+\big(\mathbf{b}.\mathbf{U}^{(2)}\big)\frac{\partial\sigma}{\partial\xi}+\left(\mathbf{c}.\frac{\partial\mathbf{U}^{(2)}}{\partial\xi}\right)\sigma+\mathbf{e}_n.\frac{\partial\mathbf{U}^{(2)}}{\partial\eta_n}\\ +\frac{\mathbf{l}[\mathbf{r}.(\nabla_{\mathbf{U}}A^k)_0]\mathbf{r}\psi_{n,k}\sigma\sigma_{,\eta_n}}{2}+\chi\sigma+\frac{\Gamma(\mathbf{l}.\mathbf{U}^{(2)}\sigma)_{,\xi}}{2}=0, \tag{27}$$

where M, $\mathbf{b}$, $\mathbf{c}$ and χ have the same meaning as in (12) with $\mathbf{e}_n$ given by

$$\mathbf{e}_n=\frac{\mathbf{l}K_n}{2}. \tag{28}$$

From (26), it follows that $\mathbf{e}_n$ is orthogonal to $\mathbf{r}$ and lies in the 4-dimensional row-space of G. Similar to the vectors $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{e}_n$ can be written as

$$\mathbf{e_n}=\beta_{n\alpha}\mathcal{G}_\alpha,\quad \alpha=1,2,3,4,\ n=1,2, \tag{29}$$

and similar to the construction of the geometric ray solution, only the projection of $\partial\mathbf{U^{(2)}}/\partial\eta_n$ onto the row space of $\mathbf{G}$ will be required to construct the term, $\mathbf{e}_n(\partial\mathbf{U^{(2)}}/\partial\eta_n)$ in the evolution equation (27). Integration of (23) gives,

$$\mathbf{G}.\mathbf{U^{(2)}}=-(K_n\mathbf{r})\int^{\xi}\frac{\partial\sigma}{\partial\eta_n}\,d\xi-\frac{1}{2}\big(\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}}A^k\big)_0\big]-\Gamma I\big)\mathbf{r}\sigma^2, \tag{30}$$

where we assume that $\mathbf{U}^{(2)} = \mathbf{0}$ when $\mathbf{U}^{(1)} = \mathbf{0}$. Differentiating the above equation with respect to η_n, we have,

$$\mathbf{G}.\frac{\partial \mathbf{U}^{(2)}}{\partial \eta_n} = -(K_n\mathbf{r})\int^{\xi} \frac{\partial^2\sigma}{\partial\eta_n\partial\eta_l}\,d\xi - \big(\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}}A^k\big)_0\big] - \Gamma I\big)\mathbf{r}\sigma\sigma_{,\eta_n}.$$

Thus the terms $\mathbf{b}.\mathbf{U}^{(2)}$, $\mathbf{c}.\partial\mathbf{U}^{(2)}/\partial\xi$ and $\mathbf{e_n}.\partial\mathbf{U}^{(2)}/\partial\eta_n$ may be constructed as follows:

$$\mathbf{b}.\mathbf{U}^{(2)} = -\boldsymbol{\omega}\left\{\frac{1}{2}\big(\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}}A_1^k\big)_0\big] - \Gamma I_1\big)\mathbf{r}\sigma^2 + (K_{1n}\mathbf{r})\int^{\xi}\frac{\partial\sigma}{\partial\eta_n}\,d\xi\right\},$$

$$\mathbf{c}.\frac{\partial\mathbf{U}^{(2)}}{\partial\xi} = -\boldsymbol{\delta}\left\{\big(\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}}A_1^k\big)_0\big] - \Gamma I_1\big)\mathbf{r}\sigma\sigma_{,\xi} + (K_{1n}\mathbf{r})\frac{\partial\sigma}{\partial\eta_n}\right\},$$

$$\mathbf{e_n}.\frac{\partial\mathbf{U}^{(2)}}{\partial\eta_n} = -\boldsymbol{\beta}_n\left\{\big(\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}}A_1^k\big)_0\big] - \Gamma I_1\big)\mathbf{r}\sigma\frac{\partial\sigma}{\partial\eta_n} + (K_{1n}\mathbf{r})\int^{\xi}\frac{\partial^2\sigma}{\partial\eta_n\partial\eta_l}\,d\xi\right\},$$

where $K_{1n} = I_1\psi_{n,t} + A_1^k\psi_{n,k}$. Substituting the above expressions in the evolution equation (27), we have,

$$\frac{d\sigma}{dt} + \left(\hat{\Gamma} + \frac{\gamma}{2}\sigma\right)\sigma\frac{\partial\sigma}{\partial\xi} + \chi\sigma + T_n\sigma\frac{\partial\sigma}{\partial\eta_n} - W_n\frac{\partial\sigma}{\partial\xi}\int^{\xi}\frac{\partial\sigma}{\partial\eta_n}\,d\xi$$
$$- M_{nl}\int^{\xi}\frac{\partial^2\sigma}{\partial\eta_n\partial\eta_l}\,d\xi = 0, \tag{31}$$

where the expressions for $\hat{\Gamma}$, γ and χ are the same as in (16), and T_n, W_n and M_{nl} are as given below:

$$T_n = -\boldsymbol{\delta}(K_{1n}\mathbf{r}) - \boldsymbol{\beta}_n\big(\phi_{,k}\big[\mathbf{r}.\big(\nabla_{\mathbf{U}}A_1^k\big)_0\big] - \Gamma I_1\big)\mathbf{r} + \frac{\mathbf{l}[\mathbf{r}.(\nabla_{\mathbf{U}}A^k)_0]\mathbf{r}\psi_{n,k}}{2},$$
$$W_n = \boldsymbol{\omega}(K_{1n}\mathbf{r}),$$
$$M_{nl} = \boldsymbol{\beta}_n(K_{1n}\mathbf{r}).$$

Equation (31) can be reduced to an equation similar to the Zabolotskaya–Khokhlov equation on differentiating with respect to ξ which is as follows:

$$\frac{\partial}{\partial\xi}\left(\frac{d\sigma}{dt} + \left(\hat{\Gamma} + \frac{\gamma}{2}\sigma\right)\sigma\frac{\partial\sigma}{\partial\xi} + \chi\sigma + \sum_{n=1}^{2}T_n\sigma\frac{\partial\sigma}{\partial\eta_n} - \sum_{n=1}^{2}W_n\frac{\partial\sigma}{\partial\xi}\int^{\xi}\frac{\partial\sigma}{\partial\eta_n}\,d\xi\right)$$
$$= \sum_{n=1}^{2}\sum_{l=1}^{2}M_{nl}\frac{\partial^2\sigma}{\partial\eta_n\partial\eta_l},$$

where $\hat{\Gamma}$, the quadratic nonlinearity coefficient may be identified with the Lax's genuine nonlinearity parameter and γ, the cubic nonlinearity coefficient, signifies the degree of material nonlinearity; the linear term $\chi\sigma$ brings about a variation in the wave amplitude as a result of the wave interactions with the varying medium ahead and changes in the geometry of the wavefront as it moves along the rays. Each of these coefficients including the additional coefficients T_n, W_n and M_{nl} which bring about the distortion of the signal in the transverse direction, is of order $O(1)$ and is a function of the space coordinates. The nonlinearity parameters lead to a distortion of the wave profile and formation of shock, while the linear term results in a growth or decay of the wave amplitude depending on whether it is negative or positive. When the medium ahead is uniform and the source term is absent, the evolution equation reduces to that of Kluwick and Cox [5] with the coefficients being absolute constants. The linear term vanishes too, if we consider one-dimensional wave propagation in the uniform medium and we get an evolution equation similar to that of Cramer and Sen [4].

5. Concluding remarks

Model equations have been derived for the propagation of high frequency pulses in a stratified atmosphere with mixed nonlinearity. The asymptotic analysis yields an inviscid Burger-like equation (16) with quadratic and cubic nonlinearities. Effects of geometric spreading and diffraction have been modelled by the perturbative approach of Kluwick and Cox. This yields a transport equation similar to the Zabolotskaya–Khokhlov equation but containing additional nonlinear terms that bring about distortion of the signal in directions transverse to that of wave propagation. In the absence of stratification, the equation reduces to that of Kluwick and Cox with all coefficients being absolute constants.

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